A Beginners' Guide to Resurgence and Trans-series in Quantum Theories

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Recent Developments in Semiclassical Probes of Quantum Field Theories
UMass Amherst ACFI, March 17-19, 2016

GD & Mithat Ünsal, reviews: 1511.05977, 1601.03414

GD, lectures at CERN 2014 Winter School

GD, lectures at Schladming 2015 Winter School

Lecture 1

- motivation: physical and mathematical
- trans-series and resurgence
- divergence of perturbation theory in QM
- basics of Borel summation
- the Bogomolny/Zinn-Justin cancellation mechanism
- towards resurgence in QFT
- effective field theory: Euler-Heisenberg effective action

Physical Motivation

• infrared renormalon puzzle in asymptotically free QFT

• non-perturbative physics without instantons: physical meaning of non-BPS saddles

• "sign problem" in finite density QFT

• exponentially improved asymptotics

Bigger Picture

• non-perturbative definition of non-trivial QFT, in the continuum

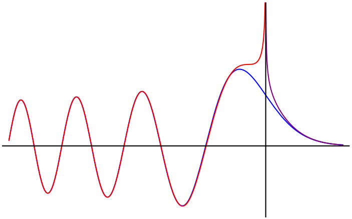
• analytic continuation of path integrals

• dynamical and non-equilibrium physics from path integrals

• uncover hidden ‘magic’ in perturbation theory

- what does a Minkowski path integral mean?

$$\int \mathcal{D}A \, \exp\left(\frac{i}{\hbar}\, S[A]\right) \quad \text{versus} \quad \int \mathcal{D}A \, \exp\left(-\frac{1}{\hbar}\, S[A]\right)$$



$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\frac{1}{3}\,t^3+x\,t\right)}\,dt \sim \begin{cases} \dfrac{e^{-\frac{2}{3}\,x^{3/2}}}{2\sqrt{\pi}\,x^{1/4}} & , \quad x \to +\infty \\[2ex] \dfrac{\sin\left(\frac{2}{3}\,(-x)^{3/2}+\frac{\pi}{4}\right)}{\sqrt{\pi}\,(-x)^{1/4}} & , \quad x \to -\infty \end{cases}$$

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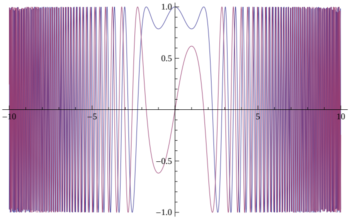

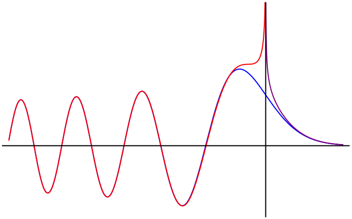

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Mathematical Motivation

Resurgence: 'new' idea in mathematics (Écalle, 1980; Stokes, 1850)

resurgence = unification of perturbation theory and non-perturbative physics

- perturbation theory generally $\Rightarrow$ divergent series
- series expansion $\longrightarrow$ *trans-series* expansion
- trans-series 'well-defined under analytic continuation'
- perturbative and non-perturbative physics entwined
- applications: ODEs, PDEs, fluids, QM, Matrix Models, QFT, String Theory, ...
- philosophical shift:

  view semiclassical expansions as potentially exact

Resurgent Trans-Series

- trans-series expansion in QM and QFT applications:

$$f(g^2) = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{k-1} \underbrace{c_{k,l,p}\, g^{2p}}_{\text{perturbative fluctuations}} \underbrace{\left(\exp\left[-\frac{c}{g^2}\right]\right)^k}_{\text{k}-\text{instantons}} \underbrace{\left(\ln\left[\pm\frac{1}{g^2}\right]\right)^l}_{\text{quasi-zero-modes}}$$

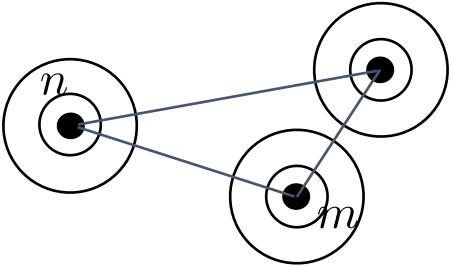
- J. Écalle (1980): closed set of functions:

(Borel transform) + (analytic continuation) + (Laplace transform)

- *trans-monomial elements*: g^2, $e^{-\frac{1}{g^2}}$, $\ln(g^2)$, are familiar
- "multi-instanton calculus" in QFT
- new: analytic continuation encoded in trans-series
- new: trans-series coefficients $c_{k,l,p}$ highly correlated
- new: exponentially improved asymptotics

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

J. Écalle, 1980



Perturbation theory

- perturbation theory generally $\to$ divergent series

e.g. QM ground state energy: $E = \sum_{n=0}^{\infty} c_n \, (\text{coupling})^n$

- Zeeman: $c_n \sim (-1)^n \, (2n)!$
- Stark: $c_n \sim (2n)!$
- cubic oscillator: $c_n \sim \Gamma(n + \frac{1}{2})$
- quartic oscillator: $c_n \sim (-1)^n \Gamma(n + \frac{1}{2})$
- periodic Sine-Gordon (Mathieu) potential: $c_n \sim n!$
- double-well: $c_n \sim n!$

note generic factorial growth of perturbative coefficients

Asymptotic Series vs Convergent Series

$$f(x) = \sum_{n=0}^{N-1} c_n \, (x - x_0)^n + R_N(x)$$

convergent series:

$$|R_N(x)| \to 0 \qquad , \quad N \to \infty \quad , \quad x \quad \text{fixed}$$

asymptotic series:

$$|R_N(x)| \ll |x - x_0|^N \qquad , \quad x \to x_0 \quad , \quad N \quad \text{fixed}$$

$\longrightarrow$ "optimal truncation":

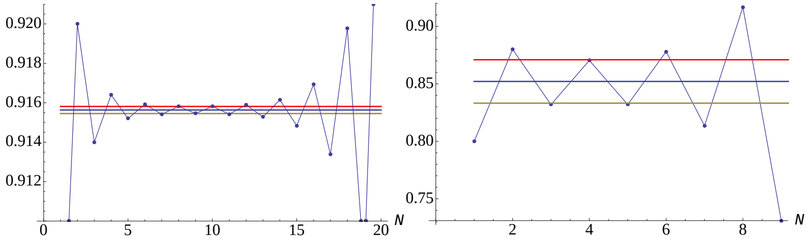
truncate just before the least term (x dependent!)

Asymptotic Series: optimal truncation & exponential precision

$$\sum_{n=0}^{\infty}(-1)^n\, n!\, x^n \sim \frac{1}{x}\, e^{\frac{1}{x}}\, E_1\left(\frac{1}{x}\right)$$

optimal truncation: $N_{\rm opt} \approx \frac{1}{x} \Rightarrow$ <u>exponentially small</u> error

$$\left|R_N(x)\right|_{N\approx 1/x} \approx \left. N!\, x^N \right|_{N\approx 1/x} \approx N! N^{-N} \approx \sqrt{N} e^{-N} \approx \frac{e^{-1/x}}{\sqrt{x}}$$



$(x = 0.1)$ $(x = 0.2)$

Borel summation: basic idea

Emile Borel

write $n! = \int_0^\infty dt\, e^{-t}\, t^n$

alternating factorially divergent series:

$$\sum_{n=0}^{\infty} (-1)^n\, n!\, g^n = \int_0^\infty dt\, e^{-t}\, \frac{1}{1+g\,t} \qquad (?)$$

integral convergent for all $g > 0$: "Borel sum" of the series

Borel Summation: basic idea

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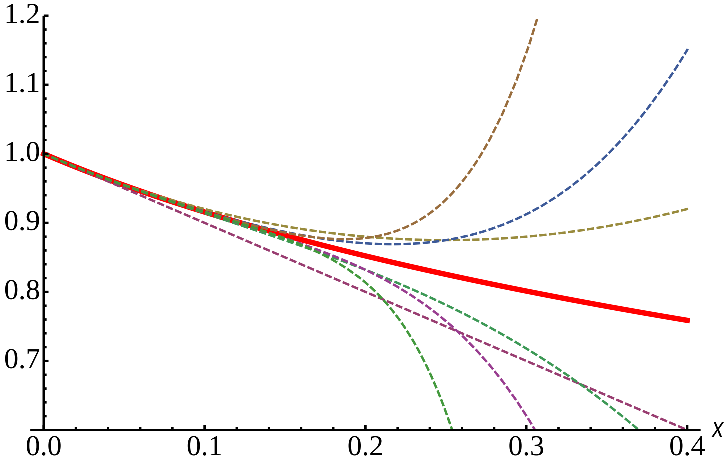

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pole on the Borel axis!

Emile Borel

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pole on the Borel axis!

Emile Borel

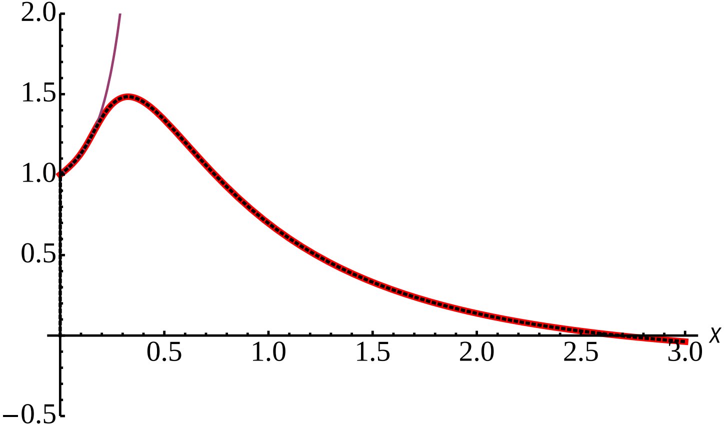
$\Rightarrow$ non-perturbative imaginary part

$$\pm \frac{i\,\pi}{g}\, e^{-\frac{1}{g}}$$

but every term in the series is real !?!

Borel Summation: basic idea

$$\text{Borel} \quad \Rightarrow \quad \mathcal{R}e\left[\sum_{n=0}^{\infty} n!\, x^n\right] = \mathcal{P}\int_0^{\infty} dt\, e^{-t}\,\frac{1}{1-x\,t} = \frac{1}{x}\, e^{-\frac{1}{x}}\, Ei\left(\frac{1}{x}\right)$$



Borel summation

Borel transform of series $f(g) \sim \sum_{n=0}^{\infty} c_n\, g^n$:

$$\mathcal{B}[f](t) = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n$$

new series typically has finite radius of convergence.

Borel resummation of original asymptotic series:

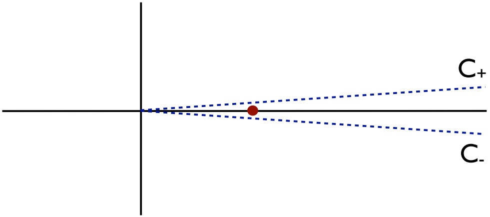
$$\mathcal{S}f(g) = \frac{1}{g} \int_0^{\infty} \mathcal{B}[f](t) e^{-t/g} dt$$

warning: $\mathcal{B}[f](t)$ may have singularities in (Borel) t plane

Borel singularities

avoid singularities on $\mathbb{R}^+$: directional Borel sums:

$$\mathcal{S}_\theta f(g) = \frac{1}{g}\int_0^{e^{i\theta}\infty} \mathcal{B}[f](t)e^{-t/g}dt$$



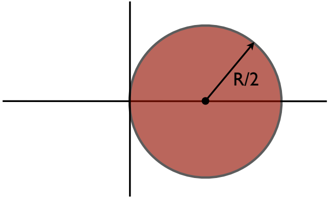
go above/below the singularity: $\theta = 0^\pm$

$\longrightarrow$ non-perturbative ambiguity: $\pm\mathrm{Im}[\mathcal{S}_0 f(g)]$

challenge: use physical input to resolve ambiguity

$f(z)$ analytic in circle $C_R = \{z : \left|z - \frac{R}{2}\right| < \frac{R}{2}\}$

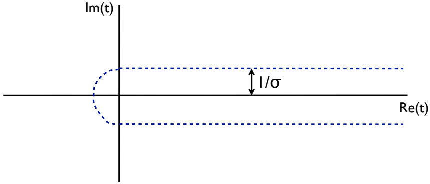
$$f(z) = \sum_{n=0}^{N-1} a_n\, z^n + R_N(z) \qquad , \qquad |R_N(z)| \leq A\, \sigma^N\, N!\, |z|^N$$



Borel transform

$$B(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\, t^n$$

analytic continuation to
$S_\sigma = \{t : |t - \mathbb{R}^+| < 1/\sigma\}$



$$f(z) = \frac{1}{z} \int_0^\infty e^{-t/z}\, B(t)\, dt$$

Borel summation in practice

$$f(g) \sim \sum_{n=0}^{\infty} c_n\, g^n \qquad , \qquad c_n \sim \beta^n\, \Gamma(\gamma\, n + \delta)$$

- alternating series: real Borel sum

$$f(g) \sim \frac{1}{\gamma} \int_0^{\infty} \frac{dt}{t} \left(\frac{1}{1+t}\right) \left(\frac{t}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\frac{t}{\beta g}\right)^{1/\gamma}\right]$$

- nonalternating series: ambiguous imaginary part

$$\mathrm{Re}\, f(-g) \;\sim\; \frac{1}{\gamma}\, \mathcal{P} \int_0^{\infty} \frac{dt}{t} \left(\frac{1}{1-t}\right) \left(\frac{t}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\frac{t}{\beta g}\right)^{1/\gamma}\right]$$

$$\mathrm{Im}\, f(-g) \;\sim\; \pm\frac{\pi}{\gamma} \left(\frac{1}{\beta g}\right)^{\delta/\gamma} \exp\left[-\left(\frac{1}{\beta g}\right)^{1/\gamma}\right]$$

another view of resurgence:

resurgence can be viewed as a method for making formal asymptotic expansions consistent with global analytic continuation properties

$\Rightarrow$ "the trans-series really IS the function"

(question: to what extent is this true/useful in physics?)

Resurgence: Preserving Analytic Continuation

- zero-dimensional partition functions

$$Z_1(\lambda) = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2\lambda}\sinh^2(\sqrt{\lambda}\,x)} = \frac{1}{\sqrt{\lambda}}\, e^{\frac{1}{4\lambda}}\, K_0\left(\frac{1}{4\lambda}\right)$$

$$\sim \sqrt{\frac{\pi}{2}}\sum_{n=0}^{\infty}(-1)^n(2\lambda)^n\frac{\Gamma(n+\frac{1}{2})^2}{n!\,\Gamma\left(\frac{1}{2}\right)^2} \qquad \text{Borel-summable}$$

$$Z_2(\lambda) = \int_{0}^{\pi/\sqrt{\lambda}} dx\, e^{-\frac{1}{2\lambda}\sin^2(\sqrt{\lambda}\,x)} = \frac{\pi}{\sqrt{\lambda}}\, e^{-\frac{1}{4\lambda}}\, I_0\left(\frac{1}{4\lambda}\right)$$

$$\sim \sqrt{\frac{\pi}{2}}\sum_{n=0}^{\infty}(2\lambda)^n\frac{\Gamma(n+\frac{1}{2})^2}{n!\,\Gamma\left(\frac{1}{2}\right)^2} \qquad \text{non-Borel-summable}$$

- naively: $Z_1(-\lambda) = Z_2(\lambda)$

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- naively: $Z_1(-\lambda) = Z_2(\lambda)$
- connection formula: $K_0(e^{\pm i\pi}\,|z|) = K_0(|z|) \mp i\,\pi\, I_0(|z|)$

$$\Rightarrow \quad Z_1(e^{\pm i\pi}\,\lambda) = Z_2(\lambda) \mp i\, e^{-\frac{1}{2\lambda}}\, Z_1(\lambda)$$

Resurgence: Preserving Analytic Continuation

- Borel summation

$$Z_1(\lambda) = \sqrt{\frac{\pi}{2}}\,\frac{1}{2\lambda}\int_0^\infty dt\, e^{-\frac{t}{2\lambda}}\; {}_2F_1\left(\frac{1}{2},\frac{1}{2},1;-t\right)$$

- directional Borel summation:

$$\begin{aligned}
&Z_1(e^{i\pi}\,\lambda) - Z_1(e^{-i\pi}\,\lambda) \\
&= \sqrt{\frac{\pi}{2}}\,\frac{1}{2\lambda}\int_1^\infty dt\, e^{-\frac{t}{2\lambda}}\left[{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;t-i\varepsilon\right) - \;{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;t+i\varepsilon\right)\right] \\
&\quad = \quad -(2i)\sqrt{\frac{\pi}{2}}\,\frac{1}{2\lambda}\,e^{-\frac{1}{2\lambda}}\int_0^\infty dt\, e^{-\frac{t}{2\lambda}}\;{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;-t\right) \\
&\quad = \quad -2\,i\,e^{-\frac{1}{2\lambda}}\,Z_1(\lambda)
\end{aligned}$$

$\left(Im\left[{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;t-i\varepsilon\right)\right] = \;{}_2F_1\left(\frac{1}{2},\frac{1}{2},1;1-t\right)\right)$

- connection formula: $Z_1(e^{\pm i\pi}\,\lambda) = Z_2(\lambda) \mp i\,e^{-\frac{1}{2\lambda}}\,Z_1(\lambda)$

Resurgence: Preserving Analytic Continuation

Stirling expansion for $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$ is divergent

$$\psi(1+z) \sim \ln z + \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \dots + \frac{174611}{6600z^{20}} - \dots$$

- functional relation: $\psi(1+z) = \psi(z) + \frac{1}{z}$

formal series $\quad \Rightarrow \quad \mathrm{Im}\,\psi(1+iy) \sim -\frac{1}{2y} + \frac{\pi}{2}$

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- reflection formula: $\psi(1+z) - \psi(1-z) = \frac{1}{z} - \pi \cot(\pi z)$

$$\Rightarrow \quad \mathrm{Im}\, \psi(1+iy) \sim -\frac{1}{2y} + \frac{\pi}{2} + \pi \sum_{k=1}^{\infty} e^{-2\pi k y}$$

"raw" asymptotics inconsistent with analytic continuation

resurgence fixes this

Transseries Example: Painlevé II (matrix models, fluids ...)

$$w'' = 2w^3(x) + x\,w(x) \qquad , \quad w \to 0 \text{ as } x \to +\infty$$

- $x \to +\infty$ asymptotics: $w \sim \sigma\, Ai(x)$

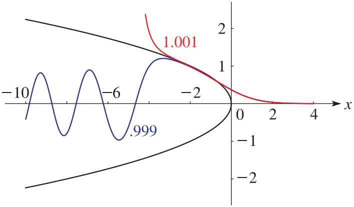
σ = real transseries parameter (flucs Borel summable)

$$w(x) \sim \sum_{n=0}^{\infty} \left(\sigma \, \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}\, x^{1/4}} \right)^{2n+1} w^{(n)}(x)$$

- $x \to -\infty$ asymptotics: $w \sim \sqrt{-\frac{x}{2}}$

transseries exponentials: $\exp\left(-\frac{2\sqrt{2}}{3}(-x)^{3/2}\right)$

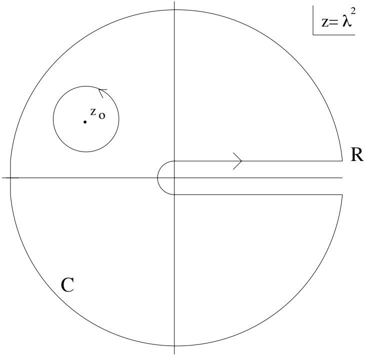
imag. part of transseries parameter fixed by cancellations



- Hastings-McLeod: $\sigma = 1$ unique real solution on $\mathbb{R}$

Borel Summation and Dispersion Relations

cubic oscillator: $V = x^2 + \lambda\, x^3$ A. Vainshtein, 1964



$$
\begin{aligned}
E(z_0) &= \frac{1}{2\pi i} \oint_C dz \, \frac{E(z)}{z - z_0} \\
&= \frac{1}{\pi} \int_0^R dz \, \frac{Im\, E(z)}{z - z_0}
\end{aligned}
$$

$$
= \sum_{n=0}^{\infty} z_0^n \left(\frac{1}{\pi} \int_0^R dz \, \frac{\mathrm{Im}\, E(z)}{z^{n+1}} \right)
$$

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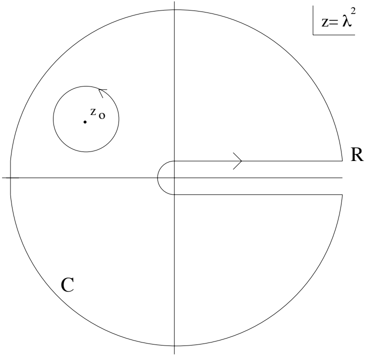

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\begin{aligned}
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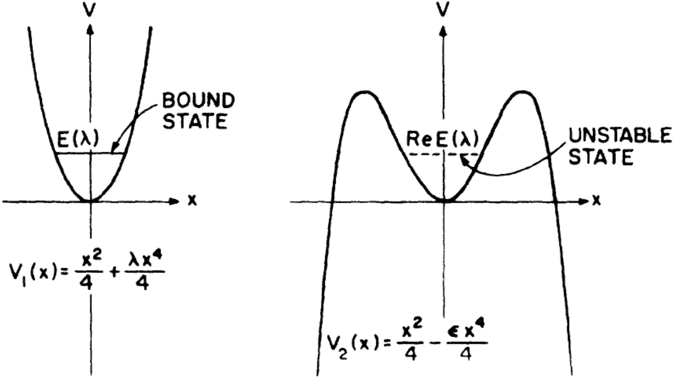
&= \sum_{n=0}^{\infty} z_0^n \left(\frac{1}{\pi}\int_0^R dz\, \frac{\mathrm{Im}\, E(z)}{z^{n+1}}\right)
\end{aligned}
$$

WKB $\Rightarrow \mathrm{Im}\, E(z) \sim \frac{a}{\sqrt{z}}\, e^{-b/z} \quad , \quad z \to 0$

$$
\Rightarrow \qquad c_n \sim \frac{a}{\pi}\int_0^\infty dz\, \frac{e^{-b/z}}{z^{n+3/2}} = \frac{a}{\pi}\,\frac{\Gamma(n+\frac{1}{2})}{b^{n+1/2}} \qquad \checkmark
$$

Instability and Divergence of Perturbation Theory

quartic AHO: $V(x) = \frac{x^2}{4} + \lambda \frac{x^4}{4}$ Bender/Wu, 1969



recall: divergence of perturbation theory in QM

e.g. ground state energy: $E = \sum_{n=0}^{\infty} c_n \, (\text{coupling})^n$

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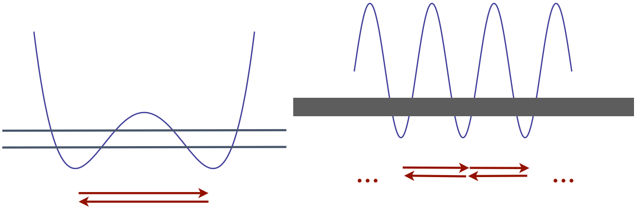
- double-well: $c_n \sim n!$

recall: divergence of perturbation theory in QM

e.g. ground state energy: $E = \sum_{n=0}^{\infty} c_n \,(\text{coupling})^n$

- Zeeman: $c_n \sim (-1)^n \,(2n)!$ stable
- Stark: $c_n \sim (2n)!$ unstable
- quartic oscillator: $c_n \sim (-1)^n \Gamma(n + \frac{1}{2})$ stable
- cubic oscillator: $c_n \sim \Gamma(n + \frac{1}{2})$ unstable
- periodic Sine-Gordon potential: $c_n \sim n!$ stable ???
- double-well: $c_n \sim n!$ stable ???

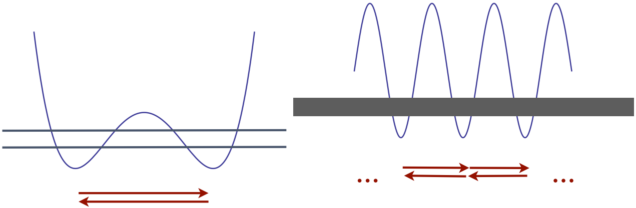
Bogomolny/Zinn-Justin mechanism in QM



- degenerate vacua: double-well, Sine-Gordon, ...

splitting of levels: a real one-instanton effect: $\Delta E \sim e^{-\frac{S}{g^2}}$

Bogomolny/Zinn-Justin mechanism in QM



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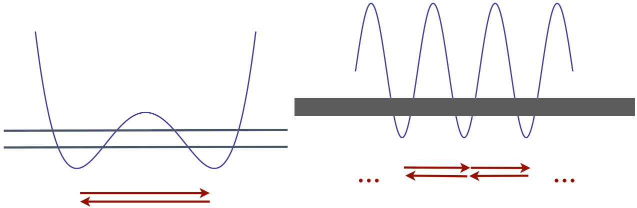
splitting of levels: a real one-instanton effect: $\Delta E \sim e^{-\frac{S}{g^2}}$

surprise: pert. theory non-Borel summable: $c_n \sim \frac{n!}{(2S)^n}$

- stable systems
- ambiguous imaginary part
- $\pm i\, e^{-\frac{2S}{g^2}}$, a 2-instanton effect

Bogomolny/Zinn-Justin mechanism in QM



- degenerate vacua: double-well, Sine-Gordon, ...

  1. perturbation theory non-Borel summable:
     ill-defined/incomplete

  2. instanton gas picture ill-defined/incomplete:
     $\mathcal{I}$ and $\bar{\mathcal{I}}$ attract

- regularize both by analytic continuation of coupling

$\Rightarrow$ ambiguous, imaginary non-perturbative terms cancel !

Bogomolny/Zinn-Justin mechanism in QM

e.g., double-well: $V(x) = x^2(1 - g\,x)^2$

$$E_0 \sim \sum_n c_n\, g^{2n}$$

- perturbation theory:

$$c_n \sim -3^n\, n! \quad : \quad \text{Borel} \quad \Rightarrow \quad \mathrm{Im}\, E_0 \sim \mp\, \pi\, e^{-\frac{1}{3g^2}}$$

- non-perturbative analysis: instanton: $g\, x_0(t) = \frac{1}{1+e^{-t}}$
- classical Eucidean action: $S_0 = \frac{1}{6g^2}$
- non-perturbative instanton gas:

$$\mathrm{Im}\, E_0 \sim \pm\, \pi\, e^{-2\frac{1}{6g^2}}$$

- BZJ cancellation $\Rightarrow$ E_0 is real and unambiguous

"resurgence" $\Rightarrow$ cancellation to all orders

Decoding of Trans-series

$$f(g^2) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{q=0}^{k-1} c_{n,k,q}\, g^{2n} \left[\exp\left(-\frac{S}{g^2}\right)\right]^k \left[\ln\left(-\frac{1}{g^2}\right)\right]^q$$

- perturbative fluctuations about vacuum: $\sum_{n=0}^{\infty} c_{n,0,0}\, g^{2n}$
- divergent (non-Borel-summable): $c_{n,0,0} \sim \alpha \frac{n!}{(2S)^n}$

$\Rightarrow$ ambiguous imaginary non-pert energy $\sim \pm i\,\pi\,\alpha\, e^{-2S/g^2}$

- but $c_{0,2,1} = -\alpha$: BZJ cancellation !

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pert flucs about instanton: $e^{-S/g^2}\ \left(1 + a_1 g^2 + a_2 g^4 + \dots\right)$

divergent:

$a_n \sim \frac{n!}{(2S)^n}\,(a\ \ln n + b) \Rightarrow \pm i\, \pi\, e^{-3S/g^2}\left(a \ln \frac{1}{g^2} + b\right)$

- 3-instanton: $e^{-3S/g^2}\left[\frac{a}{2}\left(\ln\left(-\frac{1}{g^2}\right)\right)^2 + b\ \ln\left(-\frac{1}{g^2}\right) + c\right]$

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resurgence: *ad infinitum*, also sub-leading large-order terms

Towards Resurgence in QFT

QM: divergence of perturbation theory due to factorial growth of number of Feynman diagrams

QFT: new physical effects occur, due to running of couplings with momentum

- asymptotically free QFT

$\Rightarrow$ faster source of divergence: "renormalons" (IR & UV)

QFT requires a path integral interpretation

- resurgence: 'generic' feature of steepest-descents approx.
- saddles, real and complex, BPS and non-BPS

Divergence of perturbation theory in QFT

- C. A. Hurst (1952); W. Thirring (1953):
ϕ^4 perturbation theory divergent
(i) factorial growth of number of diagrams
(ii) explicit lower bounds on diagrams

- F. J. Dyson (1952):
physical argument for divergence in QED pert. theory

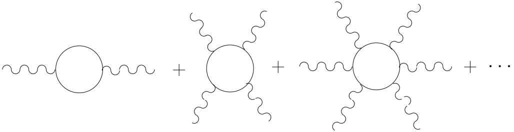
$$F(e^2) = c_0 + c_2 e^2 + c_4 e^4 + \dots$$

> *Thus [for $e^2 < 0$] every physical state is unstable against the spontaneous creation of large numbers of particles. Further, a system once in a pathological state will not remain steady; there will be a rapid creation of more and more particles, an explosive disintegration of the vacuum by spontaneous polarization.*

- *suggests* perturbative expansion cannot be convergent

Euler-Heisenberg Effective Action (1935)

review: hep-th/0406216



- 1-loop QED effective action in uniform emag field
- the birth of *effective field theory*

$$L = \frac{\vec{E}^2 - \vec{B}^2}{2} + \frac{\alpha}{90\pi}\frac{1}{E_c^2}\left[\left(\vec{E}^2 - \vec{B}^2\right)^2 + 7\left(\vec{E}\cdot\vec{B}\right)^2\right] + \dots$$

- encodes nonlinear properties of QED/QCD vacuum

> *the electromagnetic properties of the vacuum can be described by a field-dependent electric and magnetic polarisability of empty space, which leads, for example, to refraction of light in electric fields or to a scattering of light by light*
>
> *V. Weisskopf, 1936*

QFT Application: Euler-Heisenberg 1935

Folgerungen aus der Diracschen Theorie des Positrons.

Von **W. Heisenberg** und **H. Euler** in Leipzig.

Mit 2 Abbildungen. (Eingegangen am 22. Dezember 1935.)

Aus der Diracschen Theorie des Positrons folgt, da jedes elektromagnetische Feld zur Paarerzeugung neigt, eine Abänderung der Maxwellschen Gleichungen des Vakuums. Diese Abänderungen werden für den speziellen Fall berechnet, in dem keine wirklichen Elektronen und Positronen vorhanden sind, und in dem sich das Feld auf Strecken der Compton-Wellenlänge nur wenig ändert. Es ergibt sich für das Feld eine Lagrange-Funktion:

$$\mathfrak{L} = \frac{1}{2}(\mathfrak{E}^2 - \mathfrak{B}^2) + \frac{e^2}{hc}\int_0^\infty e^{-\eta}\frac{d\eta}{\eta^3}\left\{ i\eta^2(\mathfrak{E}\mathfrak{B})\cdot\frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\right)+\text{konj}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\right)-\text{konj}} + |\mathfrak{E}_k|^2 + \frac{\eta^2}{3}(\mathfrak{B}^2-\mathfrak{E}^2)\right\}.$$

($\mathfrak{E}$, $\mathfrak{B}$ Kraft auf das Elektron.

$|\mathfrak{E}_k| = \frac{m^2c^3}{e\hbar} = \frac{1}{„137“}\frac{e}{(e^2/mc^2)^2} =$ „Kritische Feldstärke“.)

- Borel transform of a (doubly) asymptotic series
- resurgent trans-series: analytic continuation $B \longleftrightarrow E$
- EH effective action $\sim$ Barnes function $\sim \int \ln\Gamma(x)$

Euler-Heisenberg Effective Action: e.g., constant B field

$$S=-\frac{B^2}{8\pi^2}\int_0^\infty \frac{ds}{s^2}\left(\coth s-\frac{1}{s}-\frac{s}{3}\right)\exp\left[-\frac{m^2 s}{B}\right]$$

$$S=-\frac{B^2}{2\pi^2}\sum_{n=0}^{\infty}\frac{\mathcal{B}_{2n+4}}{(2n+4)(2n+3)(2n+2)}\left(\frac{2B}{m^2}\right)^{2n+2}$$

- characteristic factorial divergence

$$c_n=\frac{(-1)^{n+1}}{8}\sum_{k=1}^{\infty}\frac{\Gamma(2n+2)}{(k\,\pi)^{2n+4}}$$

- reconstruct correct Borel transform:

$$\sum_{k=1}^{\infty}\frac{s}{k^2\pi^2(s^2+k^2\pi^2)}=-\frac{1}{2s^2}\left(\coth s-\frac{1}{s}-\frac{s}{3}\right)$$

Euler-Heisenberg Effective Action and Schwinger Effect

B field: QFT analogue of Zeeman effect

E field: QFT analogue of Stark effect

$B^2 \to -E^2$: series becomes non-alternating

Borel summation $\Rightarrow \mathrm{Im}\, S = \frac{e^2 E^2}{8\pi^3} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\left[-\frac{k\, m^2 \pi}{eE}\right]$

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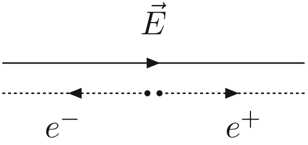
Schwinger effect:



$$2eE \frac{\hbar}{mc} \sim 2mc^2$$

$$\Rightarrow$$

$$E_c \sim \frac{m^2 c^3}{e\hbar} \approx 10^{16} \mathrm{V/cm}$$

WKB tunneling from Dirac sea

$\operatorname{Im} S \to$ physical pair production rate

- Euler-Heisenberg series must be divergent

QED/QCD effective action and the "Schwinger effect"

- formal definition:

$$\Gamma[A] = \ln \det \left(i \not{D} + m\right) \qquad\qquad D_\mu = \partial_\mu - i \frac{e}{\hbar c} A_\mu$$

- vacuum persistence amplitude

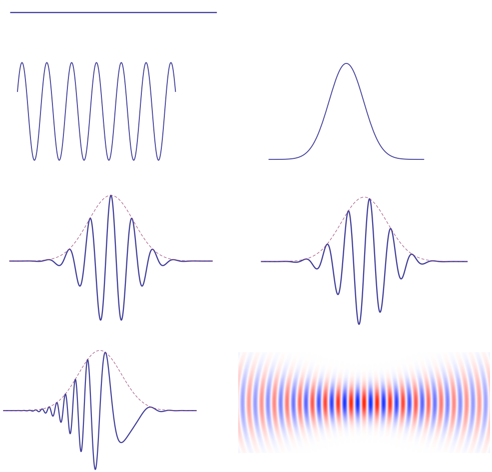
$$\langle O_{\rm out} \,|\, O_{\rm in}\rangle \equiv \exp\left(\frac{i}{\hbar}\,\Gamma[A]\right) = \exp\left(\frac{i}{\hbar}\left\{{\rm Re}(\Gamma) + i\,{\rm Im}(\Gamma)\right\}\right)$$

- encodes nonlinear properties of QED/QCD vacuum
- vacuum persistence probability

$$\left|\langle O_{\rm out} \,|\, O_{\rm in}\rangle\right|^2 = \exp\left(-\frac{2}{\hbar}\,{\rm Im}(\Gamma)\right) \approx 1 - \frac{2}{\hbar}\,{\rm Im}(\Gamma)$$

- probability of vacuum pair production $\approx \frac{2}{\hbar}\,{\rm Im}(\Gamma)$
- cf. Borel summation of perturbative series, & instantons

Schwinger Effect: Beyond Constant Background Fields



- constant field
- sinusoidal or single-pulse
- envelope pulse with sub-cycle structure; carrier-phase effect
- chirped pulse; Gaussian beam , ...

- envelopes & beyond $\Rightarrow$ complex instantons (saddles)
- physics: optimization and quantum control

Keldysh Approach in QED

Brézin/Itzykson, 1970; Popov, 1971

- Schwinger effect in $E(t) = \mathcal{E}\cos(\omega t)$
- adiabaticity parameter: $\gamma \equiv \frac{m\,c\,\omega}{e\,\mathcal{E}}$
- WKB $\quad \Rightarrow \quad P_{\rm QED} \sim \exp\left[-\pi\,\frac{m^2\,c^3}{e\,\hbar\,\mathcal{E}}\,g(\gamma)\right]$

$$P_{\rm QED} \sim \begin{cases} \exp\left[-\pi\,\frac{m^2\,c^3}{e\,\hbar\,\mathcal{E}}\right] & , \quad \gamma \ll 1 \quad \text{(non-perturbative)} \\ \\ \left(\frac{e\,\mathcal{E}}{\omega\,m\,c}\right)^{4mc^2/\hbar\omega} & , \quad \gamma \gg 1 \quad \text{(perturbative)} \end{cases}$$

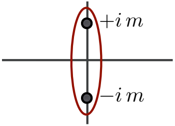
- semi-classical instanton interpolates between non-perturbative 'tunneling pair-production" and perturbative "multi-photon pair production"

Scattering Picture of Particle Production

Feynman, Nambu, Fock, Brezin/Itzykson, Marinov/Popov, ...

- over-the-barrier scattering: e.g. scalar QED

$$-\ddot{\phi} - (p_3 - e\, A_3(t))^2\, \phi = (m^2 + p_\perp^2)\phi$$



- pair production probability: $P \approx \int d^3p\, |b_p|^2$
- imaginary time method

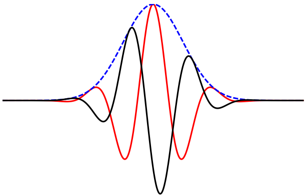
$$|b_p|^2 \approx \exp\left[-2\,\mathrm{Im} \oint dt\, \sqrt{m^2 + p_\perp^2 + (p_3 - eA_3(t))^2}\right]$$

- more structured $E(t)$ involve quantum interference

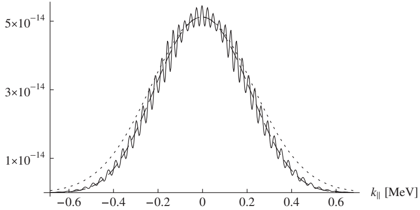
Carrier Phase Effect

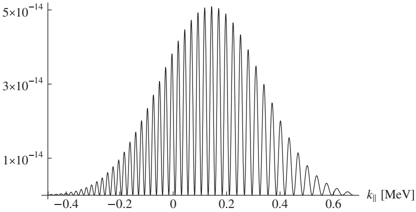
Hebenstreit, Alkofer, GD, Gies, PRL 102, 2009

$$E(t) = \mathcal{E}\ \exp\left(-\frac{t^2}{\tau^2}\right)\ \cos\left(\omega t + \varphi\right)$$



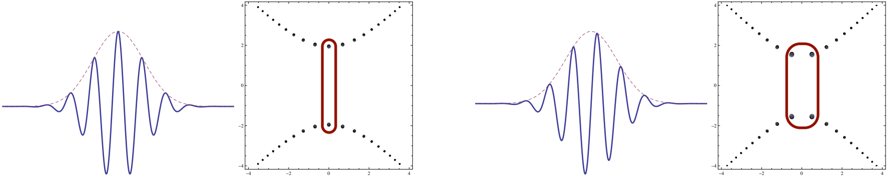
- sensitivity to carrier phase φ ?



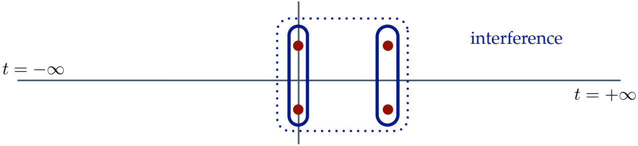


$$\varphi = 0 \qquad\qquad \varphi = \frac{\pi}{2}$$

Carrier Phase Effect from the Stokes Phenomenon



- interference produces momentum spectrum structure



$$P \approx 4 \sin^2 (\theta)\, e^{-2\,\mathrm{Im}W} \qquad\qquad \theta:\ \text{interference phase}$$

- double-slit interference, in time domain, from vacuum
- Ramsey effect: N alternating sign pulses $\Rightarrow$ N-slit system
$\Rightarrow$ coherent N^2 enhancement

Akkermans, GD, 2012

Worldline Instantons

GD, Schubert, 2005

To maintain the relativistic invariance we describe a trajectory in space-time by giving the four variables $x_\mu(u)$ as functions of some fifth parameter (somewhat analogous to the proper-time) *Feynman, 1950*

- worldline representation of effective action

$$\Gamma = -\int d^4x \int_0^\infty \frac{dT}{T} e^{-m^2 T} \oint_x \mathcal{D}x \, \exp\left[-\int_0^T d\tau \left(\dot{x}_\mu^2 + A_\mu \, \dot{x}_\mu\right)\right]$$

- double-steepest descents approximation:
- worldline instantons (saddles): $\ddot{x}_\mu = F_{\mu\nu}(x)\,\dot{x}_\nu$
- proper-time integral: $\frac{\partial S(T)}{\partial T} = -m^2$

$$\mathrm{Im}\,\Gamma \approx \sum_{\text{saddles}} e^{-S_{\text{saddle}}(m^2)}$$

- multiple turning point pairs $\Rightarrow$ complex saddles

Divergence of derivative expansion

GD, T. Hall, hep-th/9902064

- time-dependent E field: $E(t) = E\,\mathrm{sech}^2\,(t/\tau)$

$$\Gamma = -\frac{m^4}{8\pi^{3/2}}\sum_{j=0}^{\infty}\frac{(-1)^j}{(m\lambda)^{2j}}\sum_{k=2}^{\infty}(-1)^k\left(\frac{2E}{m^2}\right)^{2k}\frac{\Gamma(2k+j)\Gamma(2k+j-2)\mathcal{B}_{2k+2j}}{j!(2k)!\Gamma(2k+j+\frac{1}{2})}$$

- Borel sum perturbative expansion: large k (j fixed):

$$c_k^{(j)} \sim 2\frac{\Gamma(2k+3j-\frac{1}{2})}{(2\pi)^{2j+2k+2}}$$

$$\mathrm{Im}\,\Gamma^{(j)} \sim \exp\left[-\frac{m^2\pi}{E}\right]\frac{1}{j!}\left(\frac{m^4\pi}{4\tau^2E^3}\right)^j$$

- resum derivative expansion

$$\mathrm{Im}\,\Gamma \sim \exp\left[-\frac{m^2\pi}{E}\left(1-\frac{1}{4}\left(\frac{m}{E\tau}\right)^2+\dots\right)\right]$$

Divergence of derivative expansion

- Borel sum derivative expansion: large j (k fixed):

$$c_j^{(k)} \sim 2^{\frac{9}{2}-2k} \frac{\Gamma(2j+4k-\frac{5}{2})}{(2\pi)^{2j+2k}}$$

$$\mathrm{Im}\,\Gamma^{(k)} \sim \frac{(2\pi E\tau^2)^{2k}}{(2k)!}\, e^{-2\pi m\tau}$$

- resum perturbative expansion:

$$\mathrm{Im}\,\Gamma \sim \exp\left[-2\pi m\tau\left(1-\frac{E\tau}{m}+\dots\right)\right]$$

- compare:

$$\mathrm{Im}\,\Gamma \sim \exp\left[-\frac{m^2\pi}{E}\left(1-\frac{1}{4}\left(\frac{m}{E\tau}\right)^2+\dots\right)\right]$$

- different limits of full: $\mathrm{Im}\,\Gamma \sim \exp\left[-\frac{m^2\pi}{E}\, g\left(\frac{m}{E\tau}\right)\right]$
- derivative expansion must be divergent

Lecture 2

- uniform WKB and some magic
- resurgence from all-orders steepest descents
- towards a path integral interpretation of resurgence
- large N
- connecting weak and strong coupling
- complex saddles and quantum interference

Resurgence: recall from lecture 1

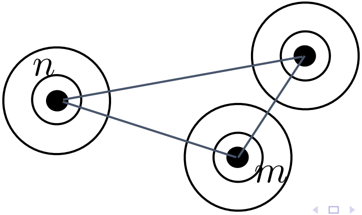
- what does a Minkowski path integral mean?

$$\int \mathcal{D}A \, \exp\left(\frac{i}{\hbar}\, S[A]\right) \quad \text{versus} \quad \int \mathcal{D}A \, \exp\left(-\frac{1}{\hbar}\, S[A]\right)$$

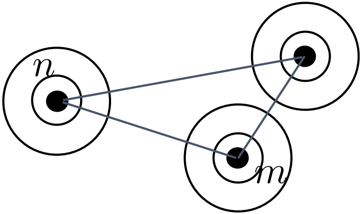
- perturbation theory is generically asymptotic

- resurgent trans-series

$$f(g^2) = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{k-1} \underbrace{c_{k,l,p}\, g^{2p}}_{\text{perturbative fluctuations}} \underbrace{\left(\exp\left[-\frac{c}{g^2}\right]\right)^k}_{\text{k}-\text{instantons}} \underbrace{\left(\ln\left[\pm\frac{1}{g^2}\right]\right)^l}_{\text{quasi-zero-modes}}$$



what is the origin of resurgent behavior in QM and QFT ?

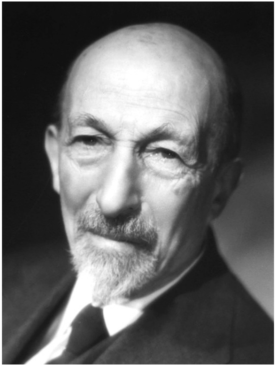


to what extent are (all?) multi-instanton effects encoded in perturbation theory? And if so, why?

- QM & QFT: basic property of all-orders steepest descents integrals
- Lefschetz thimbles: analytic continuation of path integrals

The shortest path between two truths in the real domain passes through the complex domain

Jacques Hadamard, 1865 - 1963



All-Orders Steepest Descents: Darboux Theorem

- all-orders steepest descents for contour integrals:

<u>*hyperasymptotics*</u> (Berry/Howls 1991, Howls 1992)

$$I^{(n)}(g^2) = \int_{C_n} dz\, e^{-\frac{1}{g^2} f(z)} = \frac{1}{\sqrt{1/g^2}}\, e^{-\frac{1}{g^2} f_n}\, T^{(n)}(g^2)$$

- $T^{(n)}(g^2)$: beyond the usual Gaussian approximation
- asymptotic expansion of fluctuations about the saddle n:

$$T^{(n)}(g^2) \sim \sum_{r=0}^{\infty} T_r^{(n)}\, g^{2r}$$

All-Orders Steepest Descents: Darboux Theorem

- universal resurgent relation between different saddles:

$$T^{(n)}(g^2) = \frac{1}{2\pi\, i} \sum_m (-1)^{\gamma_{nm}} \int_0^\infty \frac{dv}{v} \frac{e^{-v}}{1 - g^2 v/(F_{nm})} T^{(m)}\left(\frac{F_{nm}}{v}\right)$$

- exact resurgent relation between fluctuations about $n^{\rm th}$ saddle and about neighboring saddles m

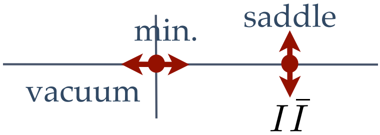
$$T_r^{(n)} = \frac{(r-1)!}{2\pi\, i} \sum_m \frac{(-1)^{\gamma_{nm}}}{(F_{nm})^r} \left[T_0^{(m)} + \frac{F_{nm}}{(r-1)} T_1^{(m)} + \frac{(F_{nm})^2}{(r-1)(r-2)} T_2^{(m)} + \dots \right.$$

- universal factorial divergence of fluctuations (Darboux)
- fluctuations about different saddles explicitly related !

$d = 0$ partition function for periodic potential $V(z) = \sin^2(z)$

$$I(g^2) = \int_0^{\pi} dz \, e^{-\frac{1}{g^2} \sin^2(z)}$$

two saddle points: $z_0 = 0$ and $z_1 = \frac{\pi}{2}$.



All-Orders Steepest Descents: Darboux Theorem

- large order behavior about saddle z_0:

$$T_r^{(0)} = \frac{\Gamma\left(r+\frac{1}{2}\right)^2}{\sqrt{\pi}\,\Gamma(r+1)}$$
$$\sim \frac{(r-1)!}{\sqrt{\pi}}\left(1 - \frac{\frac{1}{4}}{(r-1)} + \frac{\frac{9}{32}}{(r-1)(r-2)} - \frac{\frac{75}{128}}{(r-1)(r-2)(r-3)} + \right.$$

- low order coefficients about saddle z_1:

$$T^{(1)}(g^2) \sim i\sqrt{\pi}\left(1 - \frac{1}{4}g^2 + \frac{9}{32}g^4 - \frac{75}{128}g^6 + \dots\right)$$

- fluctuations about the two saddles are explicitly related could something like this work for path integrals?

"functional Darboux theorem" ?

- multi-dimensional case is already non-trivial and interesting

Pham (1965); Delabaere/Howls (2002)

- Picard-Lefschetz theory

- do a computation to see what happens ...

Resurgence in (Infinite Dim.) Path Integrals (GD, Ünsal, 1401.5202)

- periodic potential: $V(x) = \frac{1}{g^2} \sin^2(g\, x)$
- vacuum saddle point

$$c_n \sim n! \left(1 - \frac{5}{2} \cdot \frac{1}{n} - \frac{13}{8} \cdot \frac{1}{n(n-1)} - \dots \right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\, E \sim \pi\, e^{-2\frac{1}{2g^2}} \left(1 - \frac{5}{2} \cdot g^2 - \frac{13}{8} \cdot g^4 - \dots \right)$$

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$$\mathrm{Im}\, E \sim \pi\, e^{-2 \frac{1}{2g^2}} \left(1 - \frac{5}{2} \cdot g^2 - \frac{13}{8} \cdot g^4 - \dots \right)$$

- double-well potential: $V(x) = x^2(1 - gx)^2$
- vacuum saddle point

$$c_n \sim 3^n n! \left(1 - \frac{53}{6} \cdot \frac{1}{3} \cdot \frac{1}{n} - \frac{1277}{72} \cdot \frac{1}{3^2} \cdot \frac{1}{n(n-1)} - \dots \right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\, E \sim \pi\, e^{-2 \frac{1}{6g^2}} \left(1 - \frac{53}{6} \cdot g^2 - \frac{1277}{72} \cdot g^4 - \dots \right)$$

- resurgence: generic feature of differential equations
- boost invariant conformal hydrodynamics
- second-order hydrodynamics: $T^{\mu\nu} = \mathcal{E}\, u^\mu\, u^\nu + T_\perp^{\mu\nu}$

$$
\begin{aligned}
\tau \frac{d\mathcal{E}}{d\tau} &= -\frac{4}{3}\mathcal{E} + \Phi \\
\tau_{II} \frac{d\Phi}{d\tau} &= \frac{4}{3}\frac{\eta}{\tau} - \Phi - \frac{4}{3}\frac{\tau_{II}}{\tau}\Phi - \frac{1}{2}\frac{\lambda_1}{\eta^2}\Phi^2
\end{aligned}
$$

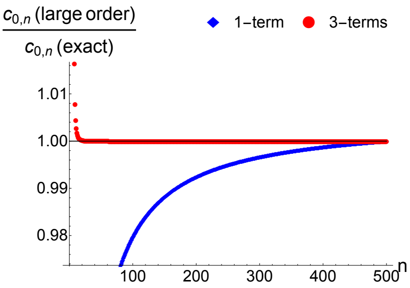
- asymptotic hydro expansion: $\mathcal{E} \sim \frac{1}{\tau^{4/3}}\left(1 - \frac{2\eta_0}{\tau^{2/3}} + \dots\right)$
- formal series $\to$ trans-series

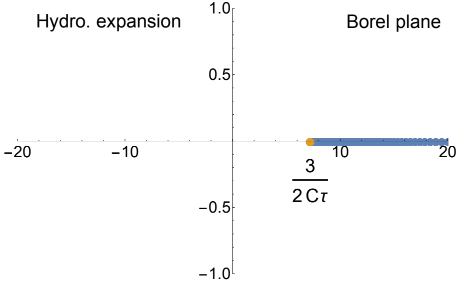
$$
\mathcal{E} \sim \mathcal{E}_{\rm pert} + e^{-S\tau^{2/3}} \times (\text{fluc}) + e^{-2S\tau^{2/3}} \times (\text{fluc}) + \dots
$$

- non-hydro modes clearly visible in the asymptotic hydro series

- study large-order behavior (Aniceto/Schiappa, 2013)

$$c_{0,k} \sim S_1 \frac{\Gamma(k+\beta)}{2\pi i\, S^{k+\beta}} \left(c_{1,0} + \frac{S\, c_{1,1}}{k+\beta-1} + \frac{S^2\, c_{1,2}}{(k+\beta-1)(k+\beta-2)} + \dots \right)$$



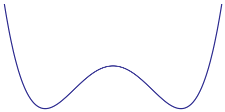


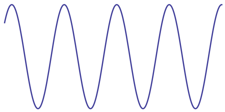
- resurgent large-order behavior and Borel structure verified to 4-instanton level

- $\Rightarrow$ trans-series for metric coefficients in AdS some magic: there is even more resurgent structure ...

Uniform WKB & Resurgent Trans-series (GD/MÜ:1306.4405, 1401.5202)

$$-\frac{\hbar^2}{2}\frac{d^2}{dx^2}\psi + V(x)\psi = E\,\psi$$





- weak coupling: degenerate harmonic classical vacua

$\Rightarrow$ uniform WKB: $\quad \psi(x) = \dfrac{D_\nu\left(\frac{1}{\sqrt{\hbar}}\varphi(x)\right)}{\sqrt{\varphi'(x)}}$

- non-perturbative effects: $\quad g^2 \leftrightarrow \hbar \quad \Rightarrow \quad \exp\left(-\frac{S}{\hbar}\right)$

- trans-series structure follows from exact quantization condition $\to E(N, \hbar) =$ trans-series

- Zinn-Justin, Voros, Pham, Delabaere, Aoki, Takei, Kawai, Koike, ...

Connecting Perturbative and Non-Perturbative Sector

Zinn-Justin/Jentschura conjecture:
generate *entire trans-series* from just two functions:

(i) perturbative expansion $E = E_{\text{pert}}(\hbar, N)$
(ii) single-instanton fluctuation function $\mathcal{P}_{\text{inst}}(\hbar, N)$
(iii) rule connecting neighbouring vacua (parity, Bloch, ...)

$$E(\hbar, N) = E_{\text{pert}}(\hbar, N) \pm \frac{\hbar}{\sqrt{2\pi}} \frac{1}{N!} \left(\frac{32}{\hbar}\right)^{N+\frac{1}{2}} e^{-S/\hbar}\, \mathcal{P}_{\text{inst}}(\hbar, N) + \ldots$$

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in fact ... (GD, Ünsal, 1306.4405, 1401.5202) fluctuation factor:

$$\mathcal{P}_{\rm inst}(\hbar, N) = \frac{\partial E_{\rm pert}}{\partial N} \exp\left[S \int_0^{\hbar} \frac{d\hbar}{\hbar^3} \left(\frac{\partial E_{\rm pert}(\hbar, N)}{\partial N} - \hbar + \frac{\left(N+\frac{1}{2}\right)\, \hbar^2}{S}\right)\right]$$

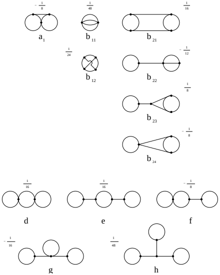
$\Rightarrow$ perturbation theory $E_{\rm pert}(\hbar, N)$ encodes everything !

Resurgence at work

- fluctuations about $\mathcal{I}$ (or $\bar{\mathcal{I}}$) saddle are determined by those about the vacuum saddle, to all fluctuation orders

- "QFT computation": 3-loop fluctuation about $\mathcal{I}$ for double-well and Sine-Gordon:

Escobar-Ruiz/Shuryak/Turbiner 1501.03993, 1505.05115



$$\mathrm{DW}: \quad e^{-\frac{S_0}{\hbar}}\left[1-\frac{71}{72}\,\hbar-0.607535\,\hbar^2-\ldots\right]$$

Resurgence at work

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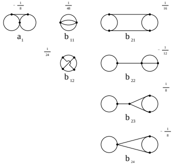

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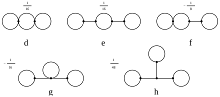


$$\text{DW}: \quad e^{-\frac{S_0}{\hbar}} \left[1 - \frac{71}{72}\,\hbar - 0.607535\,\hbar^2 - \dots \right]$$

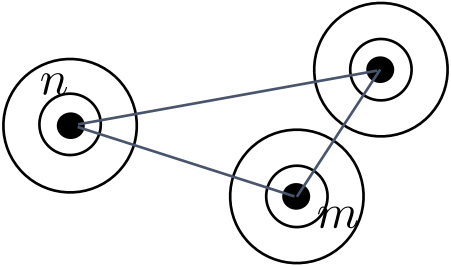
$$\text{resurgence}: e^{-\frac{S_0}{\hbar}} \left[1 + \frac{1}{72}\,\hbar \left(-102N^2 - 174N - 71\right) + \frac{1}{10368}\hbar^2 \left(10404N^4 + 17496N^3 - 2112N^2 - 14172N - 6299\right) + \dots \right]$$

- known for all N and to essentially any loop order, directly from perturbation theory !

- diagramatically mysterious ...

Connecting Perturbative and Non-Perturbative Sector

all orders of multi-instanton trans-series are encoded in perturbation theory of fluctuations about perturbative vacuum



$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{all saddles}} e^{-\frac{1}{g^2}S[A_{\text{saddle}}]} \times (\text{fluctuations}) \times (\text{qzm})$$

Analytic Continuation of Path Integrals: Lefschetz Thimbles

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{thimbles}\, k} \mathcal{N}_k\, e^{-\frac{i}{g^2}\, S_{\rm imag}[A_k]} \int_{\Gamma_k} \mathcal{D}A\, e^{-\frac{1}{g^2} S_{\rm real}[A]}$$

Lefschetz thimble = "functional steepest descents contour"

remaining path integral has real measure:

(i) Monte Carlo

(ii) semiclassical expansion

(iii) exact resurgent analysis

Analytic Continuation of Path Integrals: Lefschetz Thimbles

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{thimbles}\, k} \mathcal{N}_k\, e^{-\frac{i}{g^2}\,S_{\text{imag}}[A_k]} \int_{\Gamma_k} \mathcal{D}A\, e^{-\frac{1}{g^2}S_{\text{real}}[A]}$$

Lefschetz thimble = "functional steepest descents contour"

remaining path integral has real measure:
(i) Monte Carlo
(ii) semiclassical expansion
(iii) exact resurgent analysis

resurgence: asymptotic expansions about different saddles are closely related

requires a deeper understanding of complex configurations and analytic continuation of path integrals ...

Stokes phenomenon: intersection numbers $\mathcal{N}_k$ can change with phase of parameters

Thimbles from Gradient Flow

gradient flow to generate steepest descent thimble:

$$\frac{\partial}{\partial \tau} A(x;\tau) = -\overline{\frac{\delta S}{\delta A(x;\tau)}}$$

- keeps $Im[S]$ constant, and $Re[S]$ is monotonic

$$\frac{\partial}{\partial \tau}\left(\frac{S-\bar{S}}{2i}\right) = -\frac{1}{2i}\int\left(\frac{\delta S}{\delta A}\frac{\partial A}{\partial \tau} - \overline{\frac{\delta S}{\delta A}}\,\overline{\frac{\partial A}{\partial \tau}}\right) = 0$$

$$\frac{\partial}{\partial \tau}\left(\frac{S+\bar{S}}{2}\right) = -\int\left|\frac{\delta S}{\delta A}\right|^2$$

- Chern-Simons theory (Witten 2001)
- comparison with complex Langevin (Aarts 2013, ...)
- lattice (Tokyo/RIKEN, Aurora, 2013): Bose-gas ✓

Thimbles and Gradient Flow: an example

PHYSICAL REVIEW D **88,** 051501(R) (2013)

Monte Carlo simulations on the Lefschetz thimble: Taming the sign problem

Marco Cristoforetti,[1,2] Francesco Di Renzo,[3] Abhishek Mukherjee,[1,2] and Luigi Scorzato[1,2]
[1]*ECT*/FBK, strada delle tabarelle 286, 38123 Trento, Italy*
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(Received 31 March 2013; published 16 September 2013)

CRISTOFORETTI *et al.*

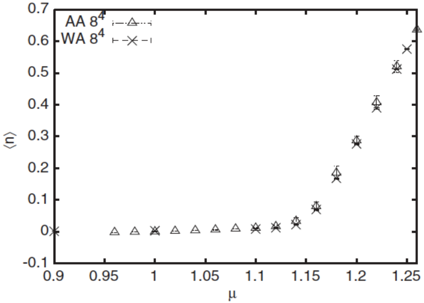
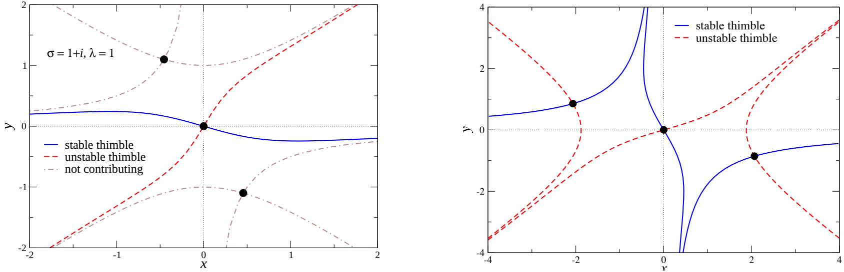


FIG. 3. Comparison of the average density $\langle n \rangle$ obtained with the worm algorithm (WA) [22] with the Aurora algorithm (AA) presented here, for the lattice $V = 8^4$. We thank C. Gattringer and T. Kloiber for providing us their results.

Thimbles, Gradient Flow and Resurgence

$$Z = \int_{-\infty}^{\infty} dx \, \exp\left[-\left(\frac{\sigma}{2}x^2 + \frac{x^4}{4}\right)\right]$$

(Aarts, 2013; GD, Unsal, ...)



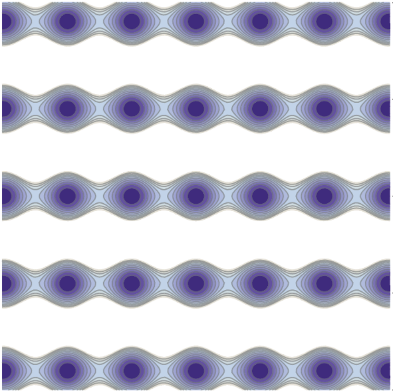
- contributing thimbles change with phase of σ
- need all three thimbles when $Re[\sigma] < 0$
- integrals along thimbles are related (resurgence)
- resurgence: preferred unique "field" choice

Ghost Instantons: Analytic Continuation of Path Integrals

(Başar, GD, Ünsal, arXiv:1308.1108)

$$\mathcal{Z}(g^2|m) = \int \mathcal{D}x\, e^{-S[x]} = \int \mathcal{D}x\, e^{-\int d\tau \left(\frac{1}{4}\dot{x}^2 + \frac{1}{g^2}\,\mathrm{sd}^2(g\,x|m)\right)}$$

- doubly periodic potential: *real* & *complex* instantons



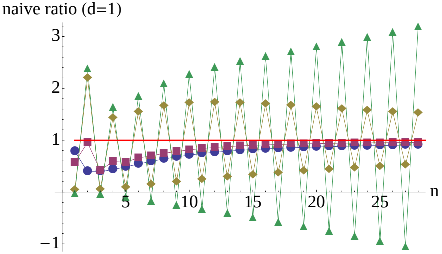
instanton actions:

$$S_{\mathcal{I}}(m) = \frac{2\arcsin(\sqrt{m})}{\sqrt{m(1-m)}}$$

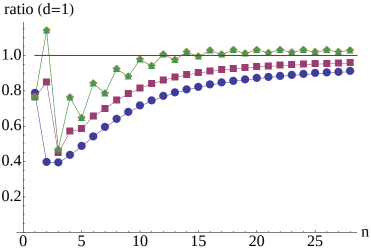
$$S_{\mathcal{G}}(m) = \frac{-2\arcsin(\sqrt{1-m})}{\sqrt{m(1-m)}}$$

- large order growth of perturbation theory:

$$a_n(m) \sim -\frac{16}{\pi} n! \left(\frac{1}{(S_{\mathcal{I}\bar{\mathcal{I}}}(m))^{n+1}} - \frac{(-1)^{n+1}}{|S_{\mathcal{G}\bar{\mathcal{G}}}(m)|^{n+1}} \right)$$



without ghost instantons



with ghost instantons

- complex instantons directly affect perturbation theory, even though they are not in the original path integral measure

Non-perturbative Physics Without Instantons

Dabrowski, GD, 1306.0921, Cherman, Dorigoni, GD, Ünsal, 1308.0127, 1403.1277

- $O(N)$ & principal chiral model have no instantons !
- Yang-Mills, $\mathbb{CP}^{N-1}$, $O(N)$, principal chiral model, ... all have non-BPS solutions with finite action

(Din & Zakrzewski, 1980; Uhlenbeck 1985; Sibner, Sibner, Uhlenbeck, 1989)

- "unstable": negative modes of fluctuation operator
- what do these mean physically ?

resurgence: ambiguous imaginary non-perturbative terms should cancel ambiguous imaginary terms coming from lateral Borel sums of perturbation theory

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{all saddles}} e^{-\frac{1}{g^2}S[A_{\text{saddle}}]} \times (\text{fluctuations}) \times (\text{qzm})$$

Connecting weak and strong coupling

main physics question:

does weak coupling analysis contain enough information to extrapolate to strong coupling ?

...even if the degrees of freedom re-organize themselves in a very non-trivial way?

classical asymptotics is clearly not enough: is resurgent asymptotics enough?

phase transitions?

Resurgence and Matrix Models, Topological Strings

Mariño, Schiappa, Weiss: *Nonperturbative Effects and the Large-Order Behavior of Matrix Models and Topological Strings* 0711.1954; Mariño, *Nonperturbative effects and nonperturbative definitions in matrix models and topological strings* 0805.3033

• resurgent Borel-Écalle analysis of partition functions etc in matrix models

$$Z(g_s, N) = \int dU \, \exp\left[\frac{1}{g_s} \mathrm{tr}\, V(U)\right]$$

• two variables: g_s and N ('t Hooft coupling: $\lambda = g_s N$)

• e.g. Gross-Witten-Wadia: $V = U + U^{-1}$

• double-scaling limit: Painlevé II

• 3rd order phase transition at $\lambda = 2$: condensation of instantons

• similar in 2d Yang-Mills on Riemann surface

Resurgence in the Gross-Witten-Wadia Model

Buividovich, GD, Valgushev 1512.09021 → PRL

- unitary matrix model ≡ 2d $U(N)$ lattice gauge theory
- third order phase transition at $\lambda = 2$

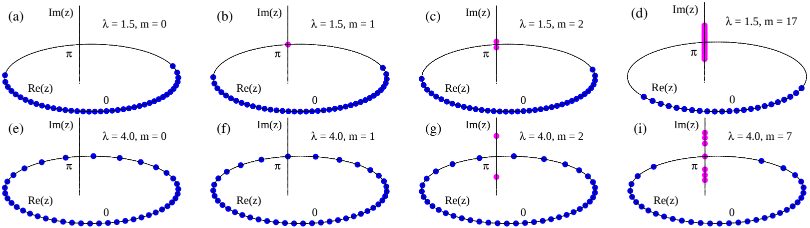
$$Z = \int \mathcal{D}U \, \exp\left[\frac{N}{\lambda}\mathrm{Tr}(U + U^\dagger)\right]$$

- in terms of eigenvalues e^{iz_i} of U

$$\begin{aligned} Z &= \prod_{i=1}^{N} \int_{-\pi}^{\pi} dz_i \, e^{-S(z_i)} \\ S(z_i) &\equiv -\frac{2N}{\lambda}\sum_i \cos(z_i) - \sum_{i<j} \ln \sin^2\left(\frac{z_i - z_j}{2}\right) \end{aligned}$$

- at large N search numerically for saddles: $\frac{\partial S}{\partial z_i} = 0$

- phase transition driven by complex saddles



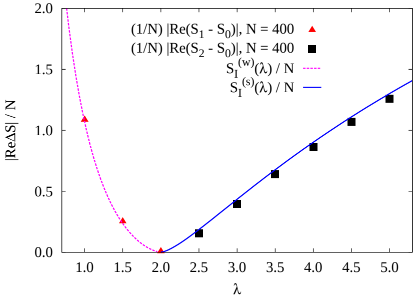
- eigenvalue tunneling into the complex plane
- weak-coupling: "instanton" is $m = 1$ configuration
  - has negative mode $\Rightarrow$ resurgent trans-series
- strong-coupling: dominant saddle is $m = 2$, complex !

- weak-coupling "instanton" action from string eqn

$$S_I^{(weak)} = 4/\lambda\sqrt{1-\lambda/2} - \text{arccosh}\left((4-\lambda)/\lambda\right), \quad \lambda < 2$$

- strong-coupling "instanton" action from string eqn

$$S_I^{(strong)} = 2\text{arccosh}\left(\lambda/2\right) - 2\sqrt{1-4/\lambda^2}, \quad \lambda \geq 2$$



- interpolated by Painlevé II (double-scaling limit)

Resurgence and Localization

(Drukker et al, 1007.3837; Mariño, 1104.0783; Aniceto, Russo, Schiappa, 1410.5834)

- certain protected quantities in especially symmetric QFTs can be reduced to matrix models $\Rightarrow$ resurgent asymptotics

- 3d Chern-Simons on $\mathbb{S}^3$ $\rightarrow$ matrix model

$$Z_{CS}(N,g)=\frac{1}{\mathrm{vol}(U(N))}\int dM\exp\left[-\frac{1}{g}\,\mathrm{tr}\left(\frac{1}{2}\left(\ln M\right)^2\right)\right]$$

- ABJM: $\mathcal{N}=6$ SUSY CS, $G=U(N)_k\times U(N)_{-k}$

$$Z_{ABJM}(N,k)=\sum_{\sigma\in S_N}\frac{(-1)^{\epsilon(\sigma)}}{N!}\int\prod_{i=1}^{N}\frac{dx_i}{2\pi k}\frac{1}{\prod_{i=1}^{N}2\mathrm{ch}\left(\frac{x_i}{2}\right)\,\mathrm{ch}\left(\frac{x_i-x_{\sigma(i)}}{2k}\right)}$$

- $\mathcal{N}=4$ SUSY Yang-Mills on $\mathbb{S}^4$

$$Z_{SYM}(N,g^2)=\frac{1}{\mathrm{vol}(U(N))}\int dM\exp\left[-\frac{1}{g^2}\,\mathrm{tr}M^2\right]$$

Connecting weak and strong coupling

- often, weak coupling expansions are divergent, but strong-coupling expansions are convergent (generic behavior for special functions)

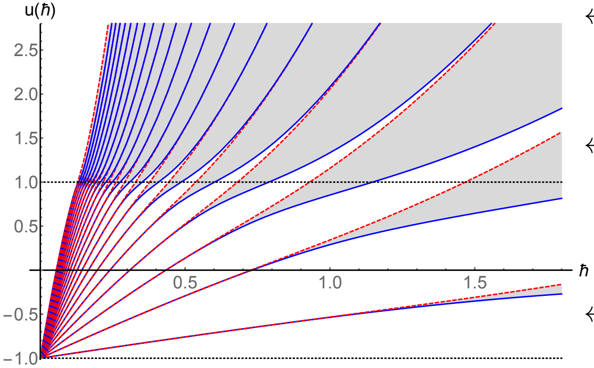
- e.g. Euler-Heisenberg

$$\Gamma(B) \sim -\frac{m^4}{8\pi^2}\sum_{n=0}^{\infty}\frac{\mathcal{B}_{2n+4}}{(2n+4)(2n+3)(2n+2)}\left(\frac{2eB}{m^2}\right)^{2n+4}$$

$$\begin{aligned}\Gamma(B) \;=\; & \frac{(eB)^2}{2\pi^2}\left\{-\frac{1}{12}+\zeta'(-1)-\frac{m^2}{4eB}+\frac{3}{4}\left(\frac{m^2}{2eB}\right)^2-\frac{m^2}{4eB}\ln(2\pi)\right.\\ & +\left[-\frac{1}{12}+\frac{m^2}{4eB}-\frac{1}{2}\left(\frac{m^2}{2eB}\right)^2\right]\ln\left(\frac{m^2}{2eB}\right)-\frac{\gamma}{2}\left(\frac{m^2}{2eB}\right)^2\\ & \left.+\frac{m^2}{2eB}\left(1-\ln\left(\frac{m^2}{2eB}\right)\right)+\sum_{n=2}^{\infty}\frac{(-1)^n\zeta(n)}{n(n+1)}\left(\frac{m^2}{2eB}\right)^{n+1}\right\}\end{aligned}$$

Resurgence in $\mathcal{N}=2$ and $\mathcal{N}=2^*$ Theories (Başar, GD, 1501.05671)

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2}+\cos(x)\,\psi=u\,\psi$$



⟵ electric sector (convergent)

⟵ magnetic sector

⟵ dyonic sector (divergent)

- energy: $u=u(N,\hbar)$; 't Hooft coupling: $\lambda\equiv N\,\hbar$
- very different physics for $\lambda\gg 1$, $\lambda\sim 1$, $\lambda\ll 1$

- Mathieu & Lamé encode Nekrasov twisted superpotential

Resurgence of $\mathcal{N}=2$ SUSY SU(2)

- moduli parameter: $u = \langle \mathrm{tr}\, \Phi^2 \rangle$
- electric: $u \gg 1$; magnetic: $u \sim 1$; dyonic: $u \sim -1$
- $a = \langle \text{scalar} \rangle$, $a_D = \langle \text{dual scalar} \rangle$, $a_D = \frac{\partial \mathcal{W}}{\partial a}$
- Nekrasov twisted superpotential $\mathcal{W}(a, \hbar, \Lambda)$:
- Mathieu equation: (Mironov/Morozov)

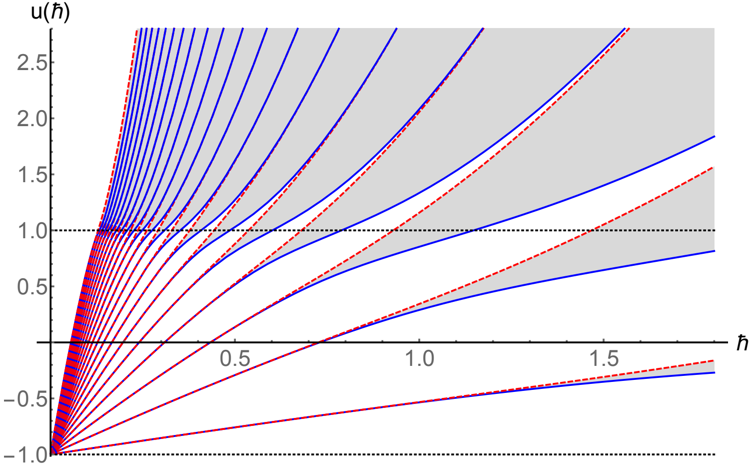
$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \Lambda^2 \cos(x)\,\psi = u\,\psi \quad , \quad a \equiv \frac{N\hbar}{2}$$

- Matone relation:

$$u(a, \hbar) = \frac{i\pi}{2}\Lambda \frac{\partial \mathcal{W}(a, \hbar, \Lambda)}{\partial \Lambda} - \frac{\hbar^2}{48}$$

Mathieu Equation Spectrum: ($\hbar$ plays role of g^2)

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = u\,\psi$$



Mathieu Equation Spectrum

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = u\,\psi$$

- small N: divergent, non-Borel-summable $\to$ trans-series

$$u(N,\hbar) \sim -1 + \hbar\left[N+\frac{1}{2}\right] - \frac{\hbar^2}{16}\left[\left(N+\frac{1}{2}\right)^2 + \frac{1}{4}\right] - \frac{\hbar^3}{16^2}\left[\left(N+\frac{1}{2}\right)^3 + \frac{3}{4}\left(N+\frac{1}{2}\right)\right] - \dots$$

- large N: convergent expansion: $\longrightarrow$?? trans-series ??

$$u(N,\hbar) \sim \frac{\hbar^2}{8}\left(N^2 + \frac{1}{2(N^2-1)}\left(\frac{2}{\hbar}\right)^4 + \frac{5N^2+7}{32(N^2-1)^3(N^2-4)}\left(\frac{2}{\hbar}\right)^8 + \frac{9N^4+58N^2+29}{64(N^2-1)^5(N^2-4)(N^2-9)}\left(\frac{2}{\hbar}\right)^{12} + \dots\right)$$

Resurgence of $\mathcal{N}=2$ SUSY SU(2) (Başar, GD, 1501.05671)

- $N\hbar \ll 1$, deep inside wells: resurgent trans-series

$$u^{(\pm)}(N,\hbar) \sim \sum_{n=0}^{\infty} c_n(N)\hbar^n \pm \frac{32}{\sqrt{\pi}\,N!}\left(\frac{32}{\hbar}\right)^{N-1/2} e^{-\frac{8}{\hbar}} \sum_{n=0}^{\infty} d_n(N)\hbar^n + \dots$$

- Borel poles at two-instanton location

- $N\hbar \gg 1$, far above barrier: convergent series

$$u^{(\pm)}(N,\hbar) = \frac{\hbar^2 N^2}{8} \sum_{n=0}^{N-1} \frac{\alpha_n(N)}{\hbar^{4n}} \pm \frac{\hbar^2}{8} \frac{\left(\frac{2}{\hbar}\right)^{2N}}{(2^{N-1}(N-1)!)^2} \sum_{n=0}^{N-1} \frac{\beta_n(N)}{\hbar^{4n}} + \dots$$

(Basar, GD, Ünsal, 2015)

- coefficients have poles at O(two-(complex)-instanton)

- $N\hbar \sim \frac{8}{\pi}$, near barrier top: "instanton condensation"

$$u^{(\pm)}(N,\hbar) \sim 1 \pm \frac{\pi}{16}\hbar + O(\hbar^2)$$

Conclusions

- Resurgence systematically unifies perturbative and non-perturbative analysis, via trans-series
- trans-series 'encode' analytic continuation information
- expansions about different saddles are intimately related
- there is extra un-tapped 'magic' in perturbation theory

- matrix models, large N, strings, SUSY QFT
- IR renormalon puzzle in asymptotically free QFT
- multi-instanton physics from perturbation theory
- $\mathcal{N} = 2$ and $\mathcal{N} = 2^*$ SUSY gauge theory

- fundamental property of steepest descents
- moral: go complex and consider all saddles, not just minima

A Few References: books

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